

Pendulum Clock Analysis and Report



Formal Report by Aaron Aviles

MAE 3 Section A02

Table of Contents

Executive Summary	3
CAD Model(Design/Assembly) Overview:	4
Theoretical Analysis	5
Point Mass Analysis:	5
1.1 CAD Model Data:	6
1.2 Measuring the Mass of the Pendulum:	7
1.3 Calculating the Total Mass of the Pendulum:	7
1.4 Calculating the Center of Mass of Pendulum:	8
Measure Center of Mass/Intermediate Verification:	9
1.5 Calculating Natural Frequency and Timing Using Point Mass Assumption:	11
Physical Measuring Time for One Revolution of Escapement Wheel:	11
1.6 Explanation for Errors of Point Mass Assumption:	12
Rigid Body Analysis:	13
General Assumptions:	13
2.1 Calculating the Total Moment of Inertia:	13
2.2 Calculating the Natural Frequency Using Rigid Body Assumption:	15
2.3 Explanation for Errors of Rigid Body Analysis	16
Discussion/Conclusion	17
Appendix:	18

Executive Summary

The MAE 3 Pendulum Clock project's objective is to design, construct, and analyze a pendulum clock utilizing a variety of CAD software and fabrication methods with the ultimate goal of comparing theoretical timing prediction of two methods, point mass assumptions and a rigid body analysis. The primary analytical objective is to measure the accuracy of each method by comparing the calculated prediction using mechanical physics principles against our experimental results. The clocks' construction of pendulum and escapement wheel are designed utilizing Autodesk Fusion 360 and cut from quarter inch blue acrylic, and assembled alongside other materials such as fasteners(bolts and nuts) for pendulum weights and to secure the clock to a stand. Experimental time data was acquired through visual timing and calculations were processed through Excel. Timing measurements were made visually with a stopwatch and resulted in 9 seconds per 1 revolution of the escapement wheel. Results show that the timing error of the point mass assumption was 20.68% while the rigid body analysis had a lower percent error of only 2.38%, illustrating that the rigid body model provides a more accurate prediction of the pendulum's natural frequency and timing.

CAD Model(Design/Assembly) Overview:

An essential part of this project is Utilization of CAD software such as fusion 360 to create our own designs. For my pendulum I chose an image that holds personal significance: the logo of my nonprofit organization, Perrybotics:

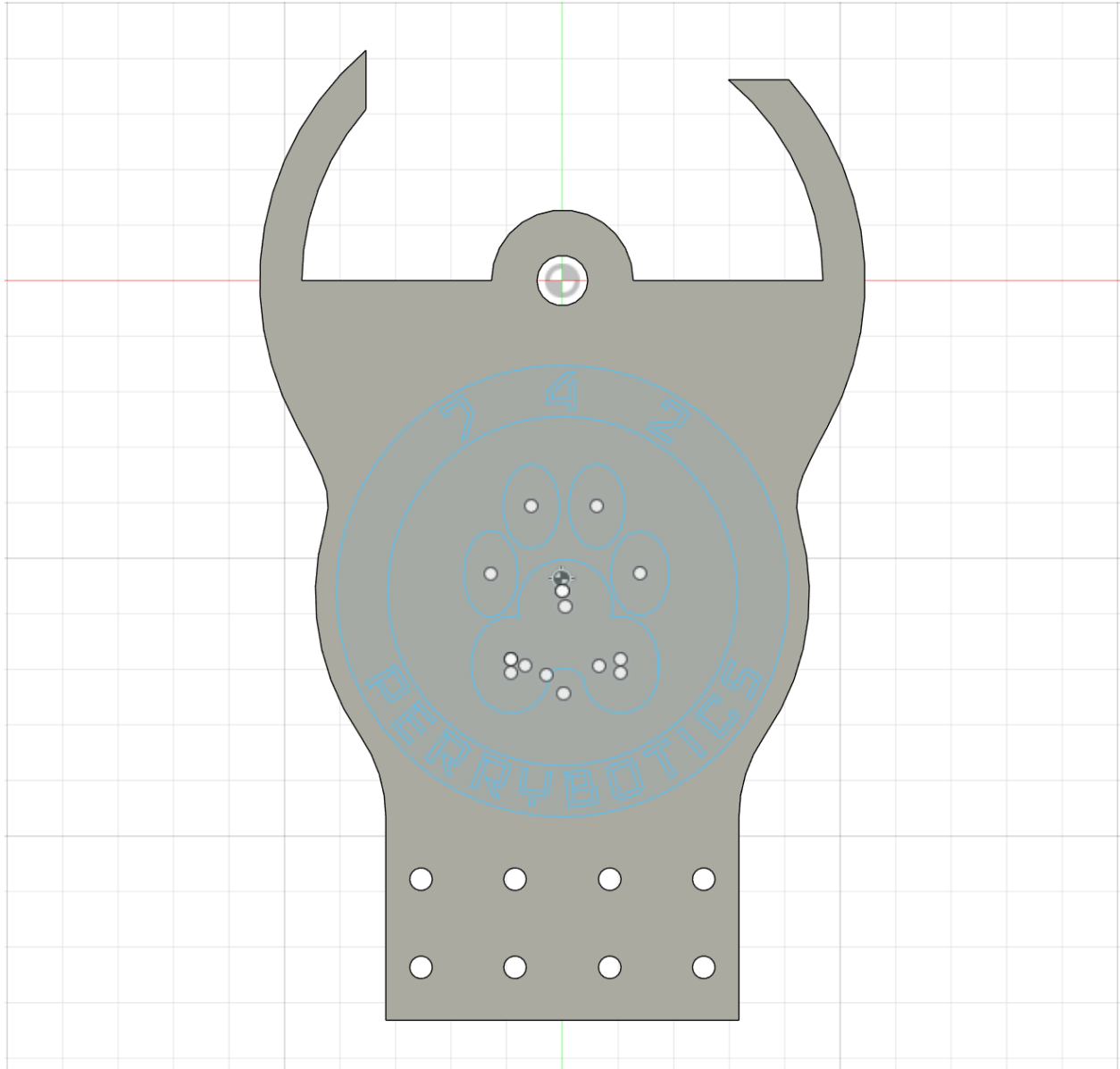


Figure 1: Top View of Pendulum_AAviles

Theoretical Analysis

This section aims to provide a detailed explanation of the methodology, physics principles, and calculations used to predict the timing of the pendulum clock. The two methods of timing analysis are **point mass assumptions**, and **rigid body assumptions**. Both of these methods aim to solve for a natural frequency of the pendulum which can be used to determine the time it takes for 1 rotation of the escapement wheel, as each full cycle of the pendulum clears one tooth, the formula for calculating time of 1 escapement revolution is as follows:

$$\text{One Revolution of Escapement Wheel} = \text{Period of oscillation} \times \text{Number of Teeth on Escapement Wheel}$$

This time calculation can then be compared to our observed time of the physical pendulum's measured time for one revolution of the escapement wheel. Video can be found: [HERE](#)

Point Mass Analysis:

Point Mass Analysis is a form of pendulum analysis that simplifies an object by assuming that all of its mass is concentrated at one point in space, this point is the center of mass. General assumptions:

- **Mass is of pendulum is concentrated at one point(at center of mass)**
- The maximum angle of the pendulum motion is relatively small(For the pendulum to be in simple harmonic motion)
- Frictional forces can be neglected
- The pendulum swings freely and the escapement wheel has no effect on the motion of the pendulum

1.1 CAD Model Data:

Utilizing Autodesk Fusion we can set the material type of the pendulum to Acrylic and derive useful data:

- Area of acrylic part: **114.273 cm²**
- Thickness of part: **0.635 cm**
- Volume: Area x Thickness = **73.198 cm³**
- Fusion's calculated mass of part: **86.96 g**
- Length to center of mass from point of rotation: **5.378 cm**

Physical	
Mass	86.960 g
Volume	73.199 cm ³
Density	1.188 g / cm ³
Area	278.948 cm ²
World X,Y,Z	0.00 cm, 0.00 cm, 0.00 cm
Center of Mass	-0.001 cm, -5.378 cm, 0.31...

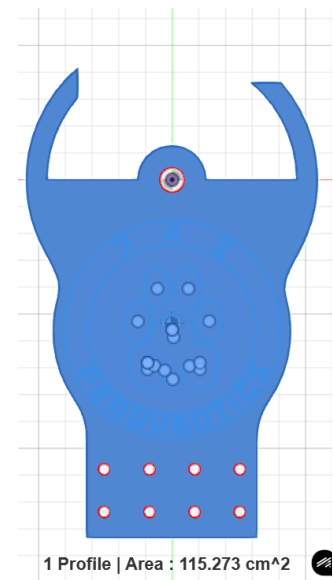


Figure 2: Physical Properties of Pendulum_AAviles

Figure 3: Area of Pendulum_AAviles

Area	A	115.273	cm ²
Thickness	t	0.635	cm
Volume	Vol	73.198	cm ³
Density	p	1.188	gm/cm ³
Calculated Mass of Acrylic	M_Calc	86.96	gm
Actual Mass of Acrylic	M_Act	72	gm
Length to Center of Mass of Acrylic	La	5.378	cm
Percent Error in Acrylic Mass Calculation	M Error	20.78%	%

Figure 4: Excel_Data_Timing Analysis_AAviles

1.2 Measuring the Mass of the Pendulum:

After importing our Fusion 360 CAD into AutoCAD and then onto lasercam to be fabricated we measured the weight of the pendulum to be 75 grams. Which utilizing this formula:

$$\text{Percent Error} = \left| \frac{M_{\text{Calc}} - M_{\text{Act}}}{M_{\text{Act}}} \right|$$

Returns a percent error of **20.78%** the source of this error will be discussed in further detail below, however we can attribute this discrepancy to manufacturing tolerances within acrylic sheet production.

1.3 Calculating the Total Mass of the Pendulum:

To calculate the Total mass of the Pendulum we must add the mass of the calculated pendulum and the mass of each bolt and nut which is approximately 4 grams per 1 bolt and 2 nuts. With a total of 8 pairs of bolt and nuts we can calculate the the theoretical total mass of the pendulum as: **118.96 grams**

Formula: $M_t = M_{\text{Calc}} + M_b \times N_b$

Calculate Total Mass of Pendulum		
Mass of One Bolt with Two Nuts	Mb	4 gm
Number of Bolts with Two Nuts	Nb	8 int
Total Mass of Pendulum with Nuts and Bolts	Mt	118.96 grams

Figure 5: Excel_Data_Timing Analysis_AAviles

1.4 Calculating the Center of Mass of Pendulum:

To calculate the Center of mass of the pendulum we summing the individual masses of each object and their length to the pivot divided by the total mass. The formula is represented as:

$$\vec{r}_{\text{c.o.m.}} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

Utilizing the dimension tool on fusion we can find the vertical distance between the pivot point and the location of each bolt and nut pair:

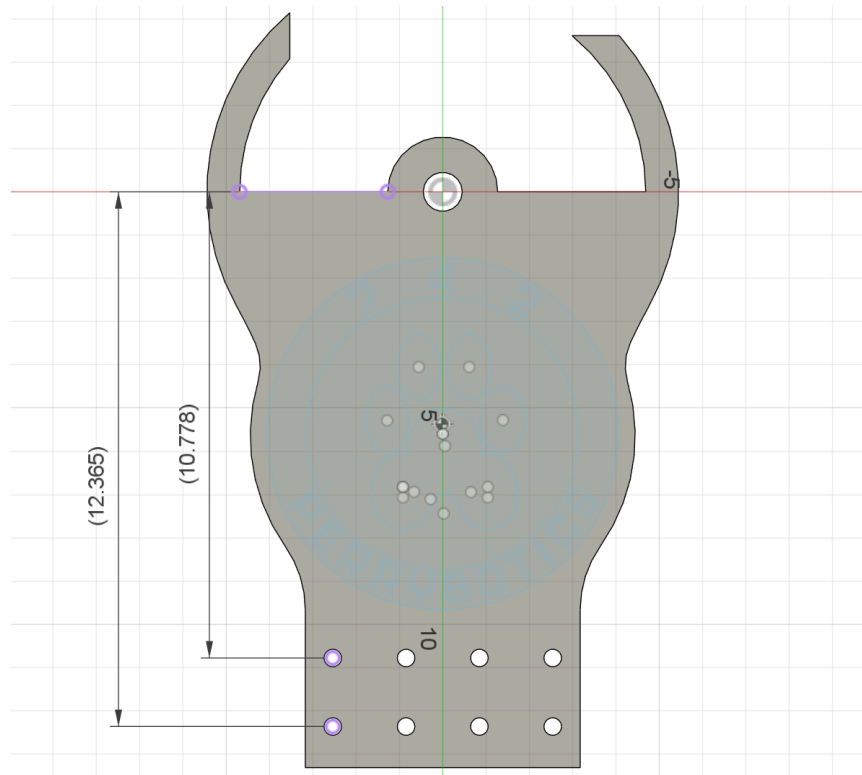


Figure 6: Dimension Sketch of Pendulum_AAviles

Virtual Measurements:

- L_bolt1 - L_bolt4 : **10.778 cm**
- L_bolt5 - L_bolt8 : **12.365 cm**

Calculated Center of Mass:

Utilizing the equation above we calculate to a length to center of mass in meter as **0.0704 m** with the imported Excel formula:

$$L_{com} = \frac{(M_{Calc} \times La) + (M_b \times (L_{bolt1} + L_{bolt2} + \dots L_{bolt8}))}{M_t}$$

Measure Center of Mass/Intermediate Verification:

To verify this calculation we must compare some physical measurements to ensure our calculations are within reasonable values. To acquire the measured center of mass, we take the pendulum and balance it, wherever the pendulum balances is its center of mass, then we can measure from the pivot to the center of mass and utilize this equation to find the percent error in our calculations. To corroborate our current location of the center of mass I balanced it on my finger and measured from the pivot the the balance point then used percent error to verify if my calculation was reasonable:

$$\text{Percent Error} = \left| \frac{\text{Lcom_est} - \text{Lcom_meter}}{\text{Lcom_meter}} \right|$$

After running measuring our balance point we find the estimated center of mass with nuts and bolts to be at **0.0736 meters** or just under 3 inches and the percent error of the length to center of mass estimated and calculated is **4.49%** a reasonable error, verifying our calculations.

Calculate Center of Mass of Pendulum with Bolts			
Length to Center of Mass of Bolt 1	L_bolt1	10.778	cm
Length to Center of Mass of Bolt 2	L_bolt2	10.778	cm
Length to Center of Mass of Bolt 3	L_bolt3	10.778	cm
Length to Center of Mass of Bolt 4	L_bolt4	10.778	cm
Length to Center of Mass of Bolt 5	L_bolt5	12.365	cm
Length to Center of Mass of Bolt 6	L_bolt6	12.365	cm
Length to Center of Mass of Bolt 7	L_bolt7	12.365	cm
Length to Center of Mass of Bolt 8	L_bolt8	12.365	cm
Length to Center of Mass in Meters	Lcom_meter	0.0704	meters
Estimated Center of Mass of Pendulum with Nuts and Bolts	Lcom_est	0.0736	meters
Percent Error in Pendulum Nuts and Bolts Lcom Estimate	Lcom_error	4.49%	percent

Figure 7: Excel_Data_Timing Analysis_AAviles

Explanation for Errors:

Small discrepancies in the calculated and measured distance to the center of mass from the pivot may come from a variety of sources, but those with the most impact are the manufacturing tolerances of the acrylic sheet as well as accuracy error with measuring the distance on a person's finger. Since these acrylic sheets are made in large batches, the resulting sheet is often very large and not uniform with thickness, with cheaper acrylic like the one we are using we can expect very large tolerances. Our sheet is specced at .25 inch thickness sheet, however when measured it really has a thickness closer to .2 inches. Because of the way we are measuring with the surface parallel to the ground, the lack of thickness affects the pendulum uniformly, which may explain why the percent error is relatively small in comparison to the errors in later calculations.

1.5 Calculating Natural Frequency and Timing Using Point Mass Assumption:

Now that we have both calculated and measured length to center of mass we can utilize the following formula to solve the natural frequency, which solves for a period of oscillation and ultimately back to our initial equation to solve for the time of 1 rotation of the escapement wheel.

Natural Frequency in Hz formula:

$$f = \frac{1}{2\pi} \sqrt{\frac{g}{L_{com}}}$$

It is important to note that g represents the constant acceleration g, **9.8 m/s²**.

Calculation Results from Excel:

- Natural frequency in radians/sec: **11.795 rad/s**
- Natural frequency in hz: **1.877 hz**
- Period: **0.532 sec/cycle**
- Time for one revolution of escapement wheel: **7.457 seconds**

Physical Measuring Time for One Revolution of Escapement Wheel:

To compare our and verify our results I measured the time it takes to complete 1 revolution of the escapement 3 times and took the average of those times to improve accuracy of stopwatch time measurement discrepancies. The measured result was **9** seconds for 1 revolution. Utilizing small modifications to the previous equation for percent error we calculate the percent error to be **20.68%**.

Calculate Natural Frequency and Timing using Point Mass Assumption			
Gravitational Constant	g	9.8	m/s^2
Natural Frequency in radians/sec	nat_freq_rad_sec	11.795	cycles/sec
Natural Frequency in Hz	nat_freq_hz	1.8773	hz
Period of Oscillation	period	0.533	sec/cycle
Number of Teeth on Escapement Wheel	nteeth	14	int
Calculated Time of One Revolution of Escapement Wheel	time_calc	7.458	sec
Measured Time of One Revolution of Escapement Wheel	time_meas	9	sec
Percent Error in Clock Timing	time_error	20.68%	percent

Figure 8: Excel_Data_Timing Analysis_AAviles

1.6 Explanation for Errors of Point Mass Assumption:

As stated above, point mass analysis assumes that mass is condensed at one point and ignores moment of inertia entirely, as to simplify the calculations. These simplifications as well as compounding errors from the calculated pendulum mass and actual pendulum mass affect the overall error. In fact the percent error of the acrylic mass calculation closely correlates to the percent error of the in clock timing, that of **20.78%** and **20.68%**. To reiterate, the manufacturing of these low tolerance sheets create massive discrepancies in the thickness of sheet, my sheet in particular being off by -0.05 inches or a thickness of just about .2 when the sheet is specced for .25 inch thickness. Which also happens to correlate to a loss of about 20% of mass.

Rigid Body Analysis:

The rigid body analysis attempts to mitigate the simplification of point mass analysis by treating the pendulum as a rigid body with distributed mass and takes into account the moment of inertia of each object including the pendulum and each bolt/nut pair.

General Assumptions:

- The maximum angle of the pendulum motion is relatively small
- Frictional forces can be neglected
- The pendulum swings freely and the escapement wheel has no effect on the motion of the pendulum

2.1 Calculating the Total Moment of Inertia:

We use Autodesk Fusion to acquire the moment of inertia of the pendulum around the pivot point denoted as Izz: **4601.686 g/cm²**

▼ Moment of Inertia at Origin (g cm ²)	
Ixx	4018.781
Ixy	0.391
Ixz	0.022
Iyx	0.391
Iyy	606.281
Iyz	148.479
Izx	0.022
Izy	148.479
Izz	4601.686

Figure 5: Moment of Inertia at Origin of Pendulum_AAviles

Additionally using fusion we can also acquire the individual moment of inertia for each bolt

denoted by the formula: $I_{bolt} = M_b \times (L_{bolt})^2$

- I_{bolt} : moment of inertia of individual bolt
- M_b : mass of bolt and nuts
- L_{bolt} : length from pivot to bolt

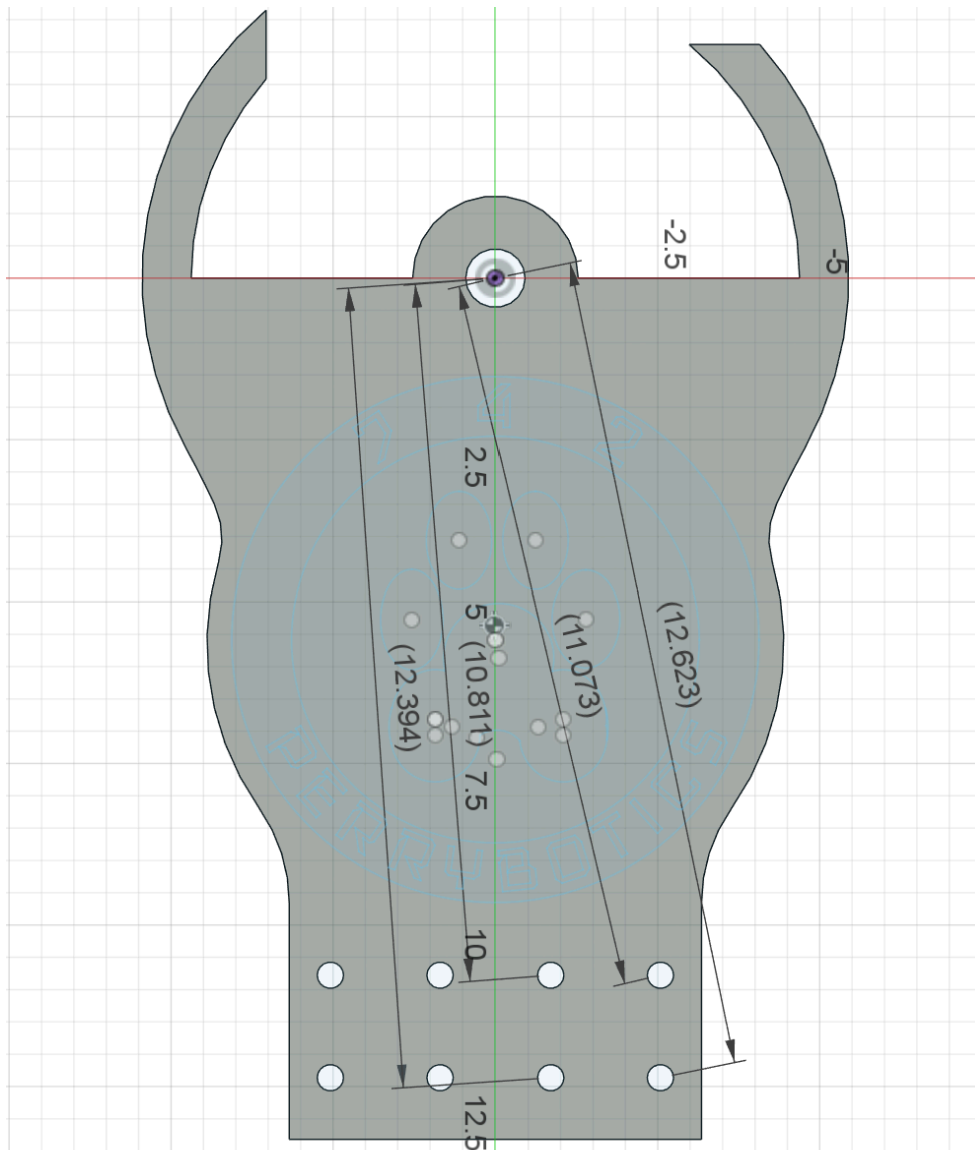


Figure 6: Sketch of Dimension at Origin of Pendulum_AAviles

Using this data we can plug into our Excel sheet and calculate the individual moment of inertia as well summate the total of moment of inertia for the system:

Moment of Inertia of Pendulum	I_a	4601.686 g/cm^2
Moment of Inertia of Bolt 1	I_bolt1	467.424 g/cm^2
Moment of Inertia of Bolt 2	I_bolt2	467.424 g/cm^2
Moment of Inertia of Bolt 3	I_bolt3	490.180 g/cm^2
Moment of Inertia of Bolt 4	I_bolt4	490.180 g/cm^2
Moment of Inertia of Bolt 5	I_bolt5	614.445 g/cm^2
Moment of Inertia of Bolt 6	I_bolt6	614.445 g/cm^2
Moment of Inertia of Bolt 7	I_bolt7	637.361 g/cm^2
Moment of Inertia of Bolt 8	I_bolt8	637.361 g/cm^2
Total Moment of Inertia	I_total	9020.505 g/cm^2

Figure 7: Excel_Data_Timing Analysis_AAviles

Moment of Inertia of Pendulum	I_a	2086.786 g/cm^2
Moment of Inertia of Bolt 1	I_bolt1	(Mb)*((10.81)^2) g/cm^2
Moment of Inertia of Bolt 2	I_bolt2	(Mb)*((10.81)^2) g/cm^2
Moment of Inertia of Bolt 3	I_bolt3	(Mb)*((11.07)^2) g/cm^2
Moment of Inertia of Bolt 4	I_bolt4	(Mb)*((11.07)^2) g/cm^2
Moment of Inertia of Bolt 5	I_bolt5	(Mb)*((12.394)^2) g/cm^2
Moment of Inertia of Bolt 6	I_bolt6	(Mb)*((12.394)^2) g/cm^2
Moment of Inertia of Bolt 7	I_bolt7	(Mb)*((12.623)^2) g/cm^2
Moment of Inertia of Bolt 8	I_bolt8	(Mb)*((12.623)^2) g/cm^2
Total Moment of Inertia	I_total	SUM(C45:C53) g/cm^2

Figure 8: Excel_Formula_Timing Analysis_AAviles

The total moment of inertia is calculated to be **9020.50 g/cm²**.

2.2 Calculating the Natural Frequency Using Rigid Body Assumption:

To calculate the natural Frequency of the pendulum using rigid body assumptions we utilize the following formula to calculate in hz:

$$f = \frac{1}{2\pi} \sqrt{\frac{M_t \times g \times L_{com}}{I_{total}}}$$

- Mt: Total mass of pendulum with nuts and bolts
- G: gravitational constant 9.8 m/s²
- Lcom: length to center mass

Results:

- Natural frequency in radians: **9.541 rad/s**
- Natural frequency in Hz: **1.518 hz**
- Period of oscillation (1/Natural frequency): **0.658 seconds/cycle**
- Calculated Time of one Revolution: **9.219 seconds**
- Percent Error (comparing against the measured time of 9 sec): **2.38%**

Natural Frequency in radians/sec	rb_nat_freq_rad_sec	9.5413	rad/s
Natural Frequency in Hz	rb_nat_freq_hz	1.519	hz
Period of Oscillation	rb_period	0.659	second/cycle
Calculated Time of One Revolution of	rb_time_calc	9.219	sec
Percent Error in Clock Timing	rb_time_error	2.38%	percent

Figure 9: Excel_Data_Timing Analysis_AAviles

The results illustrate that the rigid body analysis has a much lower percent error of only 2.38% and is fairly accurate to the actual 9 seconds needed for one full revolution of the escapement. In comparison to the the point mass assumptions which had an error 20.68% a significant difference between the two methods of analysis clearly favoring rigid body analysis

2.3 Explanation for Errors of Rigid Body Analysis

Although the error is relatively small in comparison to the other calculations of the rigid body analysis, it is still present and is most likely explained by the a number of factors, but most likely is compounded error from the initial mass error that mentioned early, that the thickness is off by a margin of nearly 20% from a specced .25 inch sheet to s .2 inch sheet. This large change in mass can contribute to the over error of this analysis. In addition small measurement discrepancies when hand stop watching the timing of one physical rotation of the escapement wheel as well as the assumption that the escapement wheel torque plays no role in the pendulum behavior which is not true.

Discussion/Conclusion

This project's primary objective was to compare two theoretical analyses, point mass and rigid body to predict the timing and behavior of the MAE 3 pendulum clock. These theoretical calculations were to be compared against physical measurements of the clock's behavior and to determine which theoretical method was more precise, based on the result the **rigid both analysis is significantly more accurate** to predict the pendulum's timing of revolution of the escapement wheel.

The rigid body analysis which consists of assumptions that the pendulum is a rigid object with non-uniform mass with individually summated moments of inertia had a calculated time of one revolution of 9.219 seconds. The physically measured time is 9 seconds flat, resulting in a percent error of **2.38%**. When compared, the point mass analysis which assumed that all mass was concentrated at a given point resulted in a calculated time of 7.457 seconds for one revolution and a percent error of **20.68%**. It is overwhelmingly clear that the rigid body model is more accurate, this very likely due to the more physically rigorous methodology utilizing center of mass, and the moment of inertia of each object of the pendulum to determine the pendulum's period while in contact with the point mass mode which grossly oversimplifies the system by concentrating all mass at a single point, ultimately leading to the point mass assumption model to be more inaccurate and error ridden.

Additionally errors that may have occurred and compounded upon our calculations is that of the mass calculated by fusion and the mass actually measured after the fabrication of the pendulum. The mass calculated by fusion measures at 86.96 grams while the physically measured pendulum measures at 72 grams, this discrepancy of 20.76% lines up with our general explanation for error throughout the theoretical analysis of manufacturing tolerances. In

manufacturing, cheaper parts generally have a larger range for tolerances, and especially large cheap sheets of acrylic such as the one MAE 3 uses are made from one large sheet of melted plastic which is left to set, this often creates either an acrylic bed that is both uniformly lower or higher than the designed .25 inch thickness and non-uniform areas of the large sheet that have either a lower or higher thickness. For my sheet when measured with a caliper, most edges of the pendulum range from 0.2 to 0.21, nearly a 20% reduction in thickness from the specified .25 desired thickness. This loss of mass of around 20% also is also neatly correlated with the approximate percent error of the point mass assumption model which yielded a percent error of 20.68%.

Additional discrepancies may have occurred due to measurement errors, such as visual delays when recording the time it takes for the escapement wheel to make one revolution, as well as the escapement wheel dynamics which is an assumption we ignored during the calculations. As well as the assumption that there is no friction between the components, this assumption in particular could have added a significant amount of error as our components are fairly cheap and old such as the brass inserts instead of bearings. These assumptions are not accounted for in our calculations and analysis and could have been contributing factors to the remaining error in our rigid body model.

Appendix: